MATH 320 Unit 6 Exercises

Quotient Rings

Let R be a ring and $S \subseteq R$. We call S a *subring* of R if it is closed under addition and multiplication, contains 0_R, and for every $a \in S$ the solution of $a + x = 0_R$ is in S (not just in R). Let R, S be rings, and $f : R \to S$ a function. We call f a *homomorphism* if it satisfies $\forall a, b \in R,\ f(a+b) = f(a) + f(b),\ f(ab) = f(a)f(b)$. Its *image* is $Im(f) = \{f(r) : r \in R\}$ and its *kernel* is $Ker(f) = \{r \in R : f(r) = 0_S\}$. If a homomorphism is also a bijection (i.e. is surjective and injective), we call it a *isomorphism*, and say that the rings are *isomorphic*.

Let I be a subring of R. We call I an *ideal* if it also satisfies $\forall r \in R, \forall a \in I,\ ra \in I,\ ar \in I$. Let R be a commutative ring with identity, let $c \in R$. We define $(c) = \{rc : r \in R\}$, called the *principal ideal generated by c*.

Let R be a ring with ideal I, and let $a, b \in R$. We say that *a is congruent to b modulo I*, writing $a \equiv b \pmod{I}$, if $a - b \in I$. Let R be a ring with ideal I, and let $a \in R$. The *congruence (or equivalence) class of a modulo I*, written $a + I$, is the set $\{b \in R : b \equiv a \pmod{I}\}$. We define R/I to be the set of equivalence classes modulo I.

Kernel to Ideal Theorem: Let $f : R \to S$ be a homomorphism of rings. Set $K = Ker(f)$, the kernel of f. Then K is an ideal in R.

Injective Homomorphism Theorem: Let $f : R \to S$ be a homomorphism of rings. f is injective if and only if its kernel is trivial, i.e. $Ker(f) = (0_R)$.

Quotient Ring Theorem: Let R be a ring with ideal I. Then:

1. R/I is a ring, called the *quotient ring of R by I*, with operations $(a+I)+(b+I) = (a+b)+I$ and $(a+I)(b+I) = (ab)+I$; and
2. If R is commutative, so is R/I; and
3. If R has an identity, so does R/I.

Ideal to Kernel Theorem: Let I be an ideal in ring R. Consider the function $\pi : R \to R/I$ via $\pi(r) = r + I$. Then π is a surjective homomorphism with kernel I.

First Isomorphism Theorem: Let $f : R \to S$ be a surjective homorphism of rings with kernel K. Then the quotient ring R/K is isomorphic to S.

An ideal M in a ring R is *maximal* if $M \neq R$ and if J is an ideal with $M \subseteq J \subseteq R$, then $J = M$ or $J = R$.

Maximal Ideal Theorem: Let M be an ideal in R, a commutative ring with identity. Then M is maximal if and only if R/M is a field.

For Dec 4:

1. Let $n \in \mathbb{Z}$ with $n \geq 2$. Recall that $f : \mathbb{Z} \to \mathbb{Z}_n$ given by $f(a) = [a]$ is a surjective homomorphism. Find its kernel, and express it as a principal ideal in $\mathbb{Z}$.

2. Consider the function $\Theta : \mathbb{R}[x] \to \mathbb{R}$ that sends each polynomial in $\mathbb{R}[x]$ to its constant term. Prove this is a surjective homomorphism, find its kernel, and express that kernel as a principal ideal.

3. Let R, S be rings, and consider $\pi_1 : R \times S \to R$ given by $\pi_1((r, s)) = r$. Prove this is a surjective homomorphism. Find its kernel, and prove that kernel is isomorphic to S.

4. Let $f : R \to S$ be a surjective homomorphism, and let I be an ideal of R. Prove that $f(I) = \{f(a) : a \in I\}$ is an ideal in S.

For Dec. 9:

5. Prove the Kernel to Ideal Theorem. You may find Unit 5 Exercise 22 to be helpful.

6. Let R be a ring. Prove that $R/(0_R)$ is isomorphic to R, and that R/R is isomorphic to the trivial ring $\{0_R\}$.

7. Let R be a noncommutative ring with ideal I. Suppose that $ab - ba \in I$ for every $a, b \in R$. Prove that R/I is commutative.

8. We call x *idempotent* if $x^2 = x$. Let I be an ideal in a ring R. Prove that every element of R/I is idempotent, if and only if $\forall a \in R,\ a^2 - a \in I$.

For Dec. 11:

9. Prove the Injective Homomorphism Theorem.

10. Use the First Isomorphism Theorem to show that $\mathbb{Z}_{20}/(5)$ is isomorphic to $\mathbb{Z}_5$.

11. Use the First Isomorphism Theorem to prove that $\mathbb{Z}[x]/(x)$ is isomorphic to $\mathbb{Z}$.

12. Describe the ideal $(x-1)$ in $\mathbb{Z}[x]$, then apply the First Isomorphism Theorem to $\mathbb{Z}[x]/(x-1)$.

Extra:

13. Find rings R, S, an ideal I of R, and a homomorphism $f : R \to S$ where $f(I) = \{f(a) : a \in I\}$ is *NOT* an ideal in S. Compare with Problem 4.

14. Prove the Quotient Ring Theorem.

15. Let R be a ring with ideal I. Prove that every element in R/I has a square root if and only if for every $a \in R$ there is some $b \in R$ with $a - b^2 \in I$.

16. Let R be a ring with ideals I, K, with $K \subseteq I$. Prove that $I/K = \{a + K : a \in I\}$ is an ideal in the quotient ring R/K.